Variational modelling elastic tubes under pure bending

M. Ahmer Wadee

a.wadee@imperial.ac.uk

Senior Lecturer
Department of Civil & Environmental Engineering
Imperial College London
London SW7 2AZ
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Acknowledgements

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- Supported by EPSRC
Instabilities: Tubes under bending

- Under lateral loading (Snap-through)
  - deformation localized
Demonstration film

- Filmed on Location in the *Hydrodynamics Laboratory*
  - Dept of Civil & Environmental Engineering, Imperial College London.
- Starring: Hose pipe tube and Ahmer Wadee’s hands
Engineer’s bending theory

- Linear theory, small strain assumption.
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- “Plane sections remain plane” and normal to the neutral surface.
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- “Plane sections remain plane” and normal to the neutral surface.
- Euler–Bernoulli equation

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\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}.
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- Cross section remains undeformed.
The behaviour of real tubes I

The Brazier effect (homogeneous ovalization)

- Initially circular section ovalizes under increasing $M$.
- First modelled by L. G. Brazier (Civ Eng IC PhD 1927) and published in Proc. Roy. Soc. London.
- Key model characteristics:
  - Progressive destiffening of the response.
  - Ovalization assumed to occur uniformly.
The behaviour of real tubes II

Kink formation

- Reissner’s model: theoretical limiting moment.
- Formation of kinks is known to occur much sooner.
- This is a localization phenomenon: more severe post-buckling response (e.g. buckling of shells, sandwich panels and pipelines).
- Aim: to account for ovalization and kink formation together.
Progressive deformation

(b) Orientation of the undeformed tube and axes.
Progressive deformation (cont.)

(d) Small curvature

(f) Ovalization

(h) Larger curvature

(j) Localization
Consider a thin circular tube with thickness $t$, radius $r$, length $L$ made of a linear elastic material with Young’s modulus $E$ and Poisson’s ratio $\nu$.

- Loading: applied uniform moment $M$.
- Deformation: a prismatic beam bends into a circular arc (approximated by a parabola).
Model formulation (cont. . . )

- Assume separate *sway* and *tilt* modes (with non-dimensional amplitudes $q_s$ and $q_t$ respectively c.f. Euler–Bernoulli theory where $q_s \equiv q_t$).

Sway:

$$y$$

$$2r$$

$$z$$

$$W$$

Tilt:

$$y$$

$$\theta$$

$$z$$

Sway and tilt modes.
Under constant applied moment the overall deformation is

\[ W(z) = q_s z (z - L)/L \]
\[ \theta(z) = q_t (2z - L)/L \]
Definition of coordinates

- Reissner deformation: \((x, y)\) moves to \((x + \zeta, y + \eta)\).
- Additional \textit{radial} deflection \(w(z, \varphi)\).
- In-plane displacement \(\tilde{u} \equiv u(z)y/r\).
Definition of constants and dimensionless quantities

- \( C = Et \)
- \( D = \frac{Et^3}{12(1 - \nu^2)} \)
- \( \Delta = \frac{r^4C}{L^2D} \)

Curvature \( \alpha \equiv 2qt\sqrt{\Delta} \)

Applied moment \( m \equiv \frac{M}{\pi r \sqrt{CD}} \).
Assumptions of the analysis

- Small strains but small nonlinear corrections.
  - First order:
    \[
    \varepsilon_z = \frac{y + \eta}{R} \equiv \frac{2q_t(y + \eta)}{L}.
    \]

- von Kármán’s strain expression:
  \[
  \varepsilon = \varepsilon_z + \frac{\partial \tilde{u}}{\partial z} + \frac{1}{2} \left( \frac{\partial w}{\partial z} \right)^2.
  \]
Total potential energy components

There are five components of energy to consider.

- Radial deformation is decomposed into the first three Fourier cosine components:

\[ w(z, \varphi) = w_0(z) + w_1(z) \cos \varphi + w_2(z) \cos 2\varphi. \]
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- Bending energy (a dot denotes differentiation wrt \( z \)):

\[ U_b = \frac{1}{2} Et \int_{0}^{L} \int_{0}^{2\pi r} \dddot{w}^2 (y + \eta)^2 \, ds \, dz. \]
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- Membrane energy:

\[ U_m = \frac{1}{2} Et \int_0^L \int_0^{2\pi r} \varepsilon^2 \, ds \, dz. \]
Total potential energy components

- Shear strain energy:

\[ U_s = \frac{1}{2} G t \int_0^L \int_0^{2\pi r} \gamma_{yz}^2 \, ds \, dz \]

where \( G = \frac{E}{2(1+\nu)} \), \( \gamma_{yz} = (q_s - q_t) \left( 2\frac{z}{L} - 1 \right) + \frac{u}{r} \).
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- **Circumferential bending energy:**

  \[ U_{cs} = \int_0^L \int_0^{2\pi r} \frac{1}{2} D \left( \frac{\partial^2 w}{\partial s^2} \right)^2 \, ds \, dz. \]
Total potential energy components

- Work done by load:

\[
M\Theta = \int_0^L \int_0^{2\pi r} Et\varepsilon_z \dot{u} ds \, dz + \int_0^L M \left( \frac{\dot{u}}{r} + \frac{2qt}{L} \right) \, dz.
\]

- Total potential energy functional:

\[
V = U_b + U_m + U_s + U_{cs} - M \Theta.
\]
$V$ is integrated around the section: single integral wrt $z$. 
Governing equations

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- Calculus of variations is used to derive the Euler–Lagrange equations for this system:
  - 3 linked 4th-order ODEs (one for each Fourier component of $w$) and a 2nd-order ODE in $u$ plus two integral constraints for $q_s$ and $q_t$. 
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- Solutions sought with solver AUTO97 on half-interval with simple-support conditions at $z = 0$ and appropriate symmetry conditions at $z = \frac{1}{2}L$. 
Equilibrium paths

Generic equilibrium behaviour: moment vs curvature. Note: uniform ovalization path, bifurcating branches and modes.
Equilibrium paths: Comparisons

Note comparisons with Reissner and our earlier model:
Numerical solutions for steel tubes

Properties of steel tube: $E = 205 \text{kN/mm}^2$, $\nu = 0.3$, $L = 400 \text{ mm}$, $L/r = 40$, $r/t = 10$
Numerical solutions for steel tubes

3-D view of tube:
Numerical solutions for steel tubes

Variation of slenderness of length $L/r$

![Graph showing the variation of slenderness of length $L/r$. The graph has a y-axis labeled $H$ ranging from 0.0 to 0.9 and an x-axis labeled $\alpha$ ranging from 0.0 to 2.0. The graph includes several curves indicating the increasing $L/r$ as $\alpha$ increases.](image)
Equilibrium paths: Higher modes

Branches for modes 3 and 4 that appear for higher $L/r$ ratios
Numerical solutions for steel tubes

Properties of steel tube: $E = 205 \text{ kN/mm}^2$, $\nu = 0.3$, $L = 400 \text{ mm}$, $L/r = 62.5$, $r/t = 10$. 
Limit moments

Scatter plot shows Limit moment $m$ vs bending stiffness $I/L$
Properties of steel tube: \( L = 400 \text{ mm}, L/r = 40, r/t = 10 \)
Numerics: Cross sections

Midspan cross-section (dotted: end-section):
For the small number modes (1 or 2) the deformation expansion is not that realistic.
On-going work

Plenty of work needs to be done:

- Inclusion of membrane energy from radial expansion
- to penalize expansion/contraction shown previously
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  - Orthotropic materials
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- Finite element or experimental validation
Conclusions

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- Inhomogeneous (localized) buckle pattern found in elastic tubes
- Buckling moment is well below Brazier/Reissner’s predictions
- Possible applications: structural engineering, biological systems, nanotechnology